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3.—A Criterion for the Existence of Solutions of Non-linear Elliptic Boundary Value Problems. By Johanna Deuel, Swiss Institute of Technology (Department of Mathematics), Zurich, and Peter Hess, Mathematics Institute, University of Zurich. *Communicated by* Professor D. E. Edmunds

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Synopsis

By a new method it is proved that a non-linear elliptic boundary value problem of rather general type admits a weak solution lying between a given weak lower solution ϕ and a given weak upper solution $\psi \geqq \phi$.

1. Introduction

In the present note we discuss the solvability of the non-linear elliptic boundary value problem

$$\text{(D)} \qquad \begin{cases} (\mathscr{A}u)(x)+p(x, u(x), \nabla u(x))=f(x) & x\in\Omega \\ u(x)=g(x) & x\in\Gamma \end{cases}$$

in a bounded domain $\Omega\subset \mathbb{R}^N$. By $\mathscr{A}$ we denote here a (linear or quasi-linear) elliptic differential operator of second order. The functions f and g are given in Ω and on the boundary Γ of Ω, respectively. Assuming the existence of a weak *upper solution* ψ:

$$\begin{cases} \mathscr{A}\psi+p(., \psi, \nabla\psi)\geqq f & \text{in } \Omega \\ \psi\geqq g & \text{on } \Gamma \end{cases} \quad \text{(in a weak sense)},$$

and of a weak *lower solution* ϕ (reversed inequalities) with $\phi \leqq \psi$ in Ω, we prove that (D) admits a weak solution u with $\phi \leqq u \leqq \psi$.

Our result extends numerous earlier statements. In the special case of $\mathscr{A}$ being linear uniformly elliptic, with Hölder continuous coefficients, and p being Hölder continuous and independent of the first-order derivatives of u, the existence of (classical) solutions was proved by Cohen, Keller, Shampine, Laetsch, Simpson, Sattinger, Amann and others [cf. for example **1** and the references cited therein]. The basic tool in all these treatments is the maximum principle, applied in the construction of a monotone iteration scheme.

Essential progress was made by Puel [7], who announced a corresponding result still for linear $\mathscr{A}$, but with p possibly depending also on ∇u. To problem (D) and the given lower and upper solution he associates an elliptic variational inequality. In order to show that a solution of this variational inequality is actually solution of problem (D), he employs a regularity result for a specific variational inequality. For boundary conditions others than Dirichlet's, the success of Puel's method thus depends on the presence of regularity results for appropriate classes of variational inequalities.

In a previous note [3] the second author introduced a new approach which follows completely the spirit of the modern theory of linear elliptic boundary value problems (weak solutions) and does not demand any regularity assumptions at all. A refinement of these arguments now allows to treat problem (D) in the proposed full generality.

For simplicity, and in order to emphasise the new method, we restrict attention to the Dirichlet problem. Similarly one treats other boundary conditions, as well as variational inequalities [thus generalising results announced in 2]. Further, our method can be easily adapted to the study of systems and enables one to extend some of the results by Martin [6].

2. Statement of the Result

Let Ω denote a bounded domain in $\mathbb{R}^N (N \geqq 1)$ with smooth boundary Γ, and let $\mathscr{A}$ be the quasi-linear elliptic second-order differential operator in divergence form:

$$(\mathscr{A}u)(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x, u(x), \nabla u(x)), \quad \text{a.e. } x \in \Omega.$$

On the functions A_i $(i = 1, ..., N)$ the following standard conditions of Leray-Lions type are imposed [e.g. **5**, Chap. II].

(A1) Each A_i: $\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Caratheodory conditions (i.e. $A_i(x, t, \xi)$ is measurable in $x \in \Omega$ for all fixed $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in (t, ξ) for a.e. fixed x). There exist constants q: $1 < q < \infty$, $c_0 \geqq 0$ and a function

$$k_0 \in L^{q'}(\Omega)\ (q' = q/q-1)$$

such that

$$| A_i(x, t, \xi)| \leqq k_0(x) + c_0(| t |^{q-1} + | \xi |^{q-1}), \quad i = 1, ..., N,$$

for a.e. $x \in \Omega$, $\forall (t, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

(A2) $\sum_{i=1}^{N} (A_i(x, t, \xi) - A_i(x, t, \xi'))(\xi_i - \xi_i') > 0$ for a.e. $x \in \Omega$, $\forall t \in \mathbb{R}$, $\forall \xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(A3) $\sum_{i=1}^{N} A_i(x, t, \xi)\xi_i \geqq \alpha | \xi |^q$ $(\alpha > 0)$, for a.e. $x \in \Omega$, $\forall (t, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

As a consequence of (A1) the semi-linear form a:

$$a(u, v) = \sum_{i=1}^{N} \int_\Omega A_i(., u, \nabla u) \frac{\partial v}{\partial x_i}\, dx$$

is defined on $W^{1, q}(\Omega) \times W^{1, q}(\Omega)$. Let further the function

$$p: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$$

satisfy the Caratheodory conditions. Note that p is however not subject to any *a priori* growth restriction. For a function v on Ω we set

$$(Pv)(x) = p(x, v(x), \nabla v(x)), \quad x \in \Omega. \tag{1}$$

Finally let

$$f \in W^{-1, q'}(\Omega)$$

and

$$g \in W^{1-1/q, q}(\Gamma)$$

be given.

DEFINITION 1. *A function u is a weak solution of problem* (D) *provided*

$$u \in W^{1,q}(\Omega), \quad u/\Gamma = g \quad \text{in} \quad W^{1-1/q,\, q}(\Gamma),$$

$$Pu \in L^{q'}(\Omega),$$

$$a(u, v) + \int_\Omega Puv dx = (f, v) \quad \forall v \in W_0^{1,q}(\Omega).$$

[We use here the notations of **5**, in particular u/Γ denotes the trace of the function u.]
A natural extension of the classical concept of upper solution is given in

DEFINITION 2. *We say that a function ψ is a weak upper solution of problem* (D) *if*

$$\psi \in W^{1,q}(\Omega), \quad \psi/\Gamma \geqq g \quad \text{in} \quad W^{1-1/q,q}(\Gamma),$$

$$P\psi \in L^{q'}(\Omega),$$

$$a(\psi, v) + \int_\Omega P\psi v dx \geqq (f, v) \quad \forall v \in W_0^{1,q}(\Omega) \quad \text{with} \quad v \geqq 0 \text{ in } \Omega.$$

Similarly a weak lower solution ϕ is characterised by the reverse inequality signs in the above definition.

THEOREM. *Suppose ϕ, ψ are weak lower and upper solution of problem* (D), *respectively, with $\phi \leqq \psi$ in Ω. Suppose further that, with a constant c_1 and a suitable function $k_1 \in L^{q'}(\Omega)$,*

$$| p(x, t, \xi)| \leqq k_1(x) + c_1 \mid \xi \mid^{q-1} \tag{2}$$

for a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^N$, $\forall t$: $\phi(x) \leqq t \leqq \psi(x)$.

Then problem (D) *admits a weak solution u with $\phi \leqq u \leqq \psi$ in Ω.*

3. PROOF OF THE THEOREM

We first associate to problem (D) and the given functions ϕ, ψ a *coercive* boundary value problem (D′), obtained by modifying the coefficient functions outside the ‘interesting’ range $\{v: \phi \leqq v \leqq \psi\}$. We then show that any solution u of problem (D′) satisfies $\phi \leqq u \leqq \psi$ and thus is asserted solution of the given problem.

(i) Let $\hat{g} \in W^{1,q}(\Omega)$ denote an extension of the function g to Ω, i.e. $\hat{g}/\Gamma = g$ in $W^{1-1/q,q}(\Gamma)$. We may choose $\hat{g}$ in such a way that $\phi \leqq \hat{g} \leqq \psi$ in Ω. Performing the change of variable $u \rightsquigarrow u - \hat{g}$ we reduce the problem to the case $\hat{g} = 0$ (possibly perturbing the right-hand side of (A3) by unessential lower order terms in $\mid \xi \mid$). *We thus assume in future that $g = 0$ and $\phi \leqq 0 \leqq \psi$ in Ω*, and search for a solution

$$u \in V \equiv W_0^{1,q}(\Omega).$$

(ii) For $i = 1, \ldots, N$ let

$$\tilde{A}_i(x, t, \xi) = \begin{cases} A_i(x, \phi(x), \xi) & t < \phi(x) \\ A_i(x, t, \xi) & \phi(x) \leqq t \leqq \psi(x) \\ A_i(x, \psi(x), \xi) & \psi(x) < t \end{cases}$$

(a.e. $x \in \Omega$, $\forall (t, \xi) \in \mathbb{R} \times \mathbb{R}^N$). The functions A_i still satisfy conditions (A1–A3). Let $\tilde{\mathscr{A}}$ be the differential operator deduced from $\mathscr{A}$ by replacing the functions A_i by $\tilde{A}_i$, and let $\tilde{a}$ denote the related form

$$\tilde{a}(u, v) = \sum_{i=1}^{N} \int_\Omega \tilde{A}_i(., u, \nabla u) \frac{\partial v}{\partial x_i} dx \quad (u, v \in W^{1,q}(\Omega)).$$

(iii) For $u \in W^{1,q}(\Omega)$ let Tu be the truncated function

$$(Tu)(x) = \begin{cases} \phi(x) & u(x) < \phi(x) \\ u(x) & \phi(x) \leqq u(x) \leqq \psi(x) \\ \psi(x) & \psi(x) < u(x) \end{cases}$$

(a.e. $x \in \Omega$). It is known that $Tu \in W^{1,q}(\Omega)$ [e.g. **8**, §1]. We need the slightly stronger

LEMMA. *The truncation mapping T is bounded and continuous from $W^{1,q}(\Omega)$ to itself.*

A proof of this lemma is provided at the end of this paper.

As a consequence of (2) the mapping P [defined by (1)] is bounded and continuous from the subset $K = \{v \in W^{1,q}(\Omega): \phi \leqq v \leqq \psi\}$ of $W^{1,q}(\Omega)$ to $L^{q'}(\Omega)$ [e.g. **4**]. Thus the corresponding mapping $P \circ T$: $W^{1,q}(\Omega) \to L^{q'}(\Omega)$ is bounded and continuous. Further, an estimate of the form

$$\left| \int_\Omega (P \circ Tu) v dx \right| \leqq (c_2 + c_3 \| u \|_{1,q}^{q-1}) \| v \|_{0,q}$$

holds for all $u, v \in W^{1,q}(\Omega)$ ($\| . \|_{k,q}$ denotes the norm in $W^{k,q}(\Omega)$).

(iv) Let the function γ: $\Omega \times \mathbb{R} \to \mathbb{R}$ be given by

$$\gamma(x, t) = \begin{cases} -(-t + \phi(x))^{q-1} & t < \phi(x) \\ 0 & \phi(x) \leqq t \leqq \psi(x) \\ (t - \psi(x))^{q-1} & \psi(x) < t \end{cases}$$

(a.e. $x \in \Omega$, $\forall t \in \mathbb{R}$). It is readily verified that γ satisfies the Caratheodory conditions. Moreover $\gamma(., v) \in L^{q'}(\Omega)$ and

$$\int_\Omega \gamma(., v) v dx \geqq \| v \|_{0,q}^q - c_4 \| v \|_{0,q}^{q-1}, \quad \forall v \in L^q(\Omega).$$

(v) We now define the semi-linear form b by

$$b(u, v) = \tilde{a}(u, v) + \int_\Omega (P \circ Tu) v dx + \beta \int_\Omega \gamma(., u) v dx$$

($u, v \in V$). Here β denotes a fixed positive number which is large enough to ensure that b is *coercive*:

$$\frac{b(v, v)}{\| v \|_{1,q}} \to +\infty \quad (v \in V, \| v \|_{1,q} \to \infty).$$

By [**5**, Theorem II. 2.8] there exists $u \in V$ such that

$$b(u, v) = (f, v) \quad \forall v \in V. \tag{3}$$

This means that the modified problem

$$\text{(D')}\qquad \begin{cases} \tilde{\mathscr{A}}u + P\circ Tu + \beta\cdot\gamma(.,u) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

admits a weak solution u. To complete our proof it suffices to show that $\phi \leqq u \leqq \psi$ in Ω, since then

$$\tilde{\mathscr{A}}u = \mathscr{A}u, \quad Tu = u, \quad \gamma(.,u) = 0.$$

(vi) Let $u \in V$ be solution of (3). We prove that $u \leqq \psi$ in Ω, the proof of $\phi \leqq u$ being accomplished similarly. In (3) we set

$$v = (u-\psi)^+ = \max\{u-\psi, 0\} \quad (\in V):$$

$$b(u, (u-\psi)^+) = (f, (u-\psi)^+). \tag{4}$$

Since

$$\int_\Omega \gamma(.,u)(u-\psi)^+ dx = \|(u-\psi)^+\|_{0,q}^q,$$

it follows from (4) that

$$\begin{aligned}(f, (u-\psi)^+) &= \tilde{a}(u, (u-\psi)^+) + \int_\Omega (P\circ Tu)(u-\psi)^+ dx + \beta\,\|(u-\psi)^+\|_{0,q}^q \\ &= \tilde{a}(u, (u-\psi)^+) + \int_\Omega P\psi(u-\psi)^+ dx \\ &\qquad + \int_\Omega (P\circ Tu - P\psi)(u-\psi)^+ dx + \beta\,\|(u-\psi)^+\|_{0,q}^q. \end{aligned} \tag{5}$$

As an immediate consequence of the definition of T,

$$\int_\Omega (P\circ Tu - P\psi)(u-\psi)^+ dx = 0. \tag{6}$$

Since ψ is weak upper solution and $(u-\psi)^+ \geqq 0$ in Ω,

$$a(\psi, (u-\psi)^+) + \int_\Omega P\psi(u-\psi)^+ dx \geqq (f, (u-\psi)^+). \tag{7}$$

By [(5), (6) and (7)] we thus obtain

$$0 \geqq \tilde{a}(u, (u-\psi)^+) - a(\psi, (u-\psi)^+) + \beta\,\|(u-\psi)^+\|_{0,q}^q. \tag{8}$$

With the notation $\Omega_+ = \{x \in \Omega : u(x) > \psi(x)\}$,

$$\begin{aligned}&\tilde{a}(u, (u-\psi)^+) - a(\psi, (u-\psi)^+) \\ &\qquad = \int_\Omega \sum_{i=1}^N (\tilde{A}_i(.,u,\nabla u) - A_i(.,\psi,\nabla\psi))\frac{\partial(u-\psi)^+}{\partial x_i}\,dx \\ &\qquad = \int_{\Omega_+} \sum_{i=1}^N (A_i(.,\psi,\nabla u) - A_i(.,\psi,\nabla\psi))\frac{\partial(u-\psi)}{\partial x_i}\,dx \\ &\qquad \geqq 0 \quad \text{(by hypothesis A2)}.\end{aligned}$$

We conclude from (8) that

$$(u-\psi)^+ = 0 \quad \text{in} \quad L^q(\Omega);$$

hence $u \leqq \psi$ in Ω.

(vii) We now provide a proof of the lemma. The boundedness of the mapping T is clear. To prove the continuity of T, it suffices for a given sequence $\{u_n\}$ with $u_n \to u$ in $W^{1,q}(\Omega)$ to show that for a suitable subsequence, $Tu_{n_k} \to Tu$ in $W^{1,q}(\Omega)$. In the following we omit the change of notation by passing to subsequences.

Since L^q-convergence implies convergence a.e. for a subsequence, we may assume

$$\left.\begin{array}{l} u_n \to u \\ \dfrac{\partial u_n}{\partial x_i} \to \dfrac{\partial u}{\partial x_i} \end{array}\right\} \quad \text{a.e. in } \Omega, \text{ for } i = 1, \dots, N.$$

There exist further subsequences and L^q-functions w, w_i $(i = 1, \dots, N)$ such that

$$\left.\begin{array}{l} |u_n(x)| \leqq w(x) \\ \left|\dfrac{\partial u_n}{\partial x_i}(x)\right| \leqq w_i(x) \end{array}\right\} \quad \text{for a.e. } x \in \Omega, \ \forall n$$

[cf. the standard proof of completeness of L^q-spaces; e.g. **9**, pp .107–108]. Since $Tu_n \to Tu$ a.e. in Ω and

$$|(Tu_n)(x)| \leqq |u_n(x)|,$$

Lebesgue's theorem on dominated convergence guarantees that

$$Tu_n \to Tu \quad \text{in} \quad L^q(\Omega).$$

Also

$$\begin{cases} \dfrac{\partial (Tu_n)}{\partial x_i}(x) \to \dfrac{\partial (Tu)}{\partial x_i}(x) \quad (n \to \infty), \\ \left|\dfrac{\partial (Tu_n)}{\partial x_i}(x)\right| \leqq w_i(x) + \left|\dfrac{\partial \phi}{\partial x_i}(x)\right| + \left|\dfrac{\partial \psi}{\partial x_i}(x)\right| \quad \forall n, \end{cases}$$

for a.e. $x \in \Omega$ and $i = 1, \dots, N$. Thus, again by the Lebesgue theorem,

$$\frac{\partial (Tu_n)}{\partial x_i} \to \frac{\partial (Tu)}{\partial x_i} \quad \text{in } L^q(\Omega) \quad (i = 1, \dots, N).$$

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